Einstein's Summation convention: If an index (except N) is repeated in a term, sumation over it from 1 to N in implied.

Therefore, we can write

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha, \quad 1 \le i \le N \qquad (8)$$

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i, \quad 1 \le \alpha \le N \qquad (9)$$

Note that α, appears twice on the rhs of Eq. (8) and i appears twice on the rhs of Eq. (9), therefore, summation over these from 1 to N is applied in the respective Equations.

We use this convention throughout this whole discussion of tensor analysis.

It is to be noted that if an index appears only once in any term, it has a definite value (any value from 1 to N). This index is called as free index. In Eq. (8), i is free index and in Eq. (9), α is free index. Further, we drop $1 \le i \le N$ in Eq. (8) and $1 \le \alpha \le N$ in Eq. (9). this should be understood.

free index

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha$$

dummy index

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i$$

Dummy index → An index which is repeated and over which summation is implied is called a dummy index. Dummy index can be replaced by any other index which does not appear in the same term.

~~Let $a_i, b_i, c_i, d_i, 1 \le i \le N$ be four~~

Tensor continued...

In the last lecture note, we have discussed basic idea about tensor, its conventions & notations which we will follow in whole discussion of tensor analysis. We have also discussed Einstein's summation convention, dummy index, free index.

We have obtained relation (see earlier note)

$$dx^{\nu} = \frac{\partial x^{\nu}}{\partial \bar{x}^{\alpha}} d\bar{x}^{\alpha} \quad \text{--- (1)}$$

free index; dummy index

$$d\bar{x}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\nu}} dx^{\nu} \quad \text{--- (2)}$$

free index; dummy index

Let's discuss an example to clarify more about Einstein's summation convention.

Ex. Let a_i, b_i, c_i, d_i, $1 \le i \le N$, be four sets of N quantities each. Then according to the Einstein's summation convention, we have

$$a_i b_i \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_N b_N \quad \text{--- (3)}$$

and

$$a_i b_j c_j \equiv a_i b_1 c_1 + a_i b_2 c_2 + a_i b_3 c_3 + \cdots + a_i b_N c_N \quad \text{--- (4)}$$

i is free index here ~~[illegible]~~ have fixed value between 1 to N.

Eq. (3) can also be written as

$$a_i b_i = a_j b_j = a_k b_k = a_r b_r = a_l b_l \text{ , etc} \quad \text{---(5)}$$

In above equation same index is occuring twice in a term. ~~So~~ These are dummy indices.

Again

$$a_i b_j c_j = a_i b_k c_k = a_i b_l c_l \quad \text{---(6)}$$

~~i is dummy index in above equation~~

j, k, l are dummy indices in above ~~equation~~ expression which cannot be replaced by 'i' since 'i' appears in the same term.

Therefore, $$a_i b_j c_j \neq a_i b_i c_i \quad \text{---(7)}$$

The above Eq. can be verified, ~~to~~ write

$$a_i b_i c_i \equiv a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \cdots + a_N b_N c_N \quad \text{---(8)}$$

See Eq. (8) and (4) are not same. Thus, Eq. (7) is true.

Again Consider expressions $a_i b_i c_i d_i$ and $a_i b_i c_j d_j$.
Now.

$$a_i b_i c_i d_i \equiv a_1 b_1 c_1 d_1 + a_2 b_2 c_2 d_2 + a_3 b_3 c_3 d_3 + \cdots + a_N b_N c_N d_N; \quad \text{---(9)}$$

$$a_i b_i c_j d_j \equiv \left(\sum_{i=1}^{N} a_i b_i\right)\left(\sum_{j=1}^{N} c_j d_j\right)$$

$$= (a_1 b_1 + a_2 b_2 + \cdots + a_N b_N)(c_1 d_1 + c_2 d_2 + \cdots + c_N d_N) \quad \text{---(10)}$$

From Eqs. (9) and (10)

$$a_i b_i c_i d_i \neq a_i b_i c_j d_j$$

Also we can write $a_i b_i c_j d_j = a_i b_i c_k d_k = a_l b_l c_i d_i$, etc

~~Consider Eq. ① and ②, From Eq. ① we can write~~

Since coordinates x^i are independent of each other, therefore

$$\frac{dx^i}{dx^j} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j \end{cases} \quad \text{——— (11)}$$

We define the Kronecker delta symbol by

$$\delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \text{——— (12)}$$

Now Eq. (11) ~~and (12)~~ can be written as

$$\frac{dx^i}{dx^j} = \delta^i_j, \text{ —— (13)}$$

Similarly, the coordinates $\bar{x}^\alpha$ are also independent of each other, so that

$$\frac{d\bar{x}^\alpha}{d\bar{x}^\beta} = \delta^\alpha_\beta$$

If x^i are functions of $\bar{x}^\alpha$, then we can write

$$\frac{dx^i}{dx^j} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\alpha}{\partial x^j}$$

Using Eq. (11) and (12), we can write.

$$\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\alpha}{\partial x^j} = \delta^i_j \text{ ——— (14)}$$

Similarly, we obtain

$$\frac{\partial \bar{x}^\alpha}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\beta} = \delta^\alpha_\beta \text{ ——— (15)}$$